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SPATIAL GEOMETRY IN SECONDARY SCHOOL EDUCATION: TRAINING SPATIAL ABILITIES AND IDENTIFYING STUDENTS' PROBLEMS IN SPATIAL GEOMETRY

The article addresses the teaching of spatial geometry in secondary schools with the implications to geometric courses at the university. We will focus especially on geometric topics which are taught in the Czech Republic, i.e. solid geometry, cross section of solids, transformations of two- and three-dimensional shapes in three-dimensional space and many more. We will show several examples of typical geometric tasks from secondary school education and advanced topics from geometric courses at the university. Experiences gained from lecturing geometric courses (at Charles University, Czech Republic) intended for student newcomers to university show serious gap in students' spatial abilities. We will identify some of concrete students' problems in spatial geometry and introduce several approaches, newly suggested exercises and activities for the both secondary school and university students which could mitigate these gaps. Our activities in the classrooms are supported with dynamic system GeoGebra which is widely used in education process all over the world.

Keywords: spatial geometry; spatial ability; secondary school education; GeoGebra

MOTIVATION

The teaching of descriptive geometry which includes planar and spatial geometry has a very strong tradition in the Czech Republic. The Czech Republic (in the 19th and the beginning of the 20th century part of the Austria-Hungarian Empire) and the neighbor states (Austria, Slovakia, Poland, and Germany) belonged to the countries where the teaching of descriptive geometry and the research in descriptive geometry were highly developed, (Grattan-Guinness, 2016). The Czech curriculum in elementary and secondary schools still continues in the tradition of teaching geometry (we refer here mainly to Euclidean geometry) and Czech universities (for instance Charles University and Czech Technical University) with technically oriented study programs focus on the teaching of several fields of geometry (in addition to Euclidean geometry, there are taught algebraic, differential, non-Euclidean geometry, or topology). Unfortunately, the advanced level of the teaching of school geometry in the both elementary and secondary schools is on the decrease which is mainly caused by fewer hours devoted to geometry. This trend has to have repercussions on the content of geometry courses at universities. I have been teaching Euclidean geometry, descriptive geometry and computational geometry at the Faculty of Mathematics and Physics (Charles University) in the Czech Republic for eight years (plus five years before at Czech Technical University). The courses are devoted to undergraduate and graduate students who study the specialization of teaching of mathematics and descriptive geometry, i.e. the prospective secondary school teachers. The basic requirement for all these subjects is the fundamental knowledge of planar and spatial geometry. If I compared the syllabuses from the very beginning of my professional career as a university teacher, I have to admit that even in this relatively short period I had to radically reduce the complexity of my courses. There are evidently many factors which could involve it and has to be taken into consideration ranging from not sufficient secondary school education (or even elementary school education) of my students to the lack of students' interest and motivation. We have to also add that the curricula at the Faculty of Mathematics and Physics have changed and been modified during the years to follow general needs and trends in the society. The offer of taught subjects has been increased, new scientific fields and mathematical disciplines and their applications are shown to students and everything is in the tandem of modern technologies and computer software. There is no more need to study for example descriptive geometry in such way (to draw by hand very complex scenes, projections of real objects, technical devices, or buildings) as in the last century when the engineers, architects or designers didn't have modern computer software for implementation of their projects. On the other hand, to use computer graphic technologies still requires the perfect knowledge of geometrical principals. The needs moved dramatically but in my opinion the necessity of the knowledge of various fields of geometry still remains. We face new challenges in today's Information Age and many fields of geometry are of immediate relevance to them, (Pottman et al., 2007).

From the scientific point of view, the practical applications of contemporary geometric fields such as Euclidean, differential, algebraic geometry or topology include image understanding; computer vision and robot motion planning; pattern pre-processing in machine learning; and computer-aided architectural and industrial design. Geometry is essential to the manufacturing, engineering and construction industries; the video game industry (e.g. the export of real-world interiors and exteriors into the virtual worlds of computer games); reverse engineering; the digitization of real objects using 3D scanning; digital surface reconstruction from point clouds; the replication of the shapes of real-world objects using 3D printing; computer graphics and many more. See for example (LaValle, 2006; Farin, Hoschek, & Kim, 2002; Eilam, 2005; Lipson & Kurman, 2013; Foley et al., 1995). All these applications can be characterized by combinations of geometric principles.

It's not necessary to focus only on scientific fields and professional areas where geometry is essential. Geometry is even important for our everyday life. We need the spatial skills to be able to orient ourselves in the environment, to understand the spatial relations among objects, to solve everyday tasks such as packing, moving, and many more.

Because of mentioned reasons of importance of geometry in many areas there should be even stronger efforts to maintain the quality of teaching of geometry at all school levels and at universities.

SPATIAL ABILITY

In our article let's concentrate on the specific topics of geometry which could develop, train and support our spatial abilities. According to Lean and Clements (1981) *spatial ability* can be described as the ability to formulate mental images and to manipulate these images in the mind. Sorby (1999) distinguished *spatial ability* and *spatial skills*. Spatial ability is considered as the innate ability to visualize, whereas spatial skills are learned and acquired through training. Anyway, the both terms are very closely interconnected and sometimes can be hardly distinguished. Tatre (1990a) specified spatial skills as mental skills which are concerned with understanding, manipulating, reorganizing, or interpreting relationships visually. Tatre also described two different major types of spatial skills based on several research materials in educational psychology mainly on (McGee, 1979). The categorization of spatial skills is based on mental processes which are used for solving certain tasks and two major categories are *spatial visualization* and *spatial orientation*. These two categories depend on what is being moved while we solve some tasks. If we need to move or to alter in our mind some parts or the whole mentally presented objects, then it is considered as a spatial visualization task. On the contrary, a spatial orientation task does not consider mentally moving the presented object. The mental picture of an object or a situation remains the same and fixed but the task requires the mental movement of our viewpoint. According to Tatre (1990a) spatial orientation tasks could involve organizing, recognizing, making sense out of a visual representation, seeing it from a different angle, but not mentally moving the object. The more detailed classification of spatial skills is presented in (Tatre, 1990b.) It should be pointed out that some authors introduce even additional major types of spatial skills. For more details, see (Katsioloudis, Jovanovic, & Jones, 2014).

There are several empirical studies in which the researchers attempt to find the appropriate methods for improving the students' spatial abilities. Katsioloudis, Jovanovic, & Jones (2014) summarize these methods. Let's mention those which I follow in my geometry courses - the spatial abilities could be improved through certain instructional designs, using computer-based 3D visualizations, through activities predominantly consisting of free-hand sketching and object manipulation.

This paper is focused on the identifying geometric tasks which require spatial orientation and visualization skills and which are problematic for students. We also suggest activities which could support and improve students' spatial skills which are considered to be important in technically oriented education, (Sorby, 2009).

SPATIAL GEOMETRY

The scope of our interest is school geometry, mainly spatial geometry. In the following section let's mention the main geometric topics which I need my students to know very well when they come to the university and we build on these topics in Euclidean geometry, descriptive geometry and computational geometry courses at the faculty. The majority of students who come to the Faculty of Mathematics and Physics graduated from the grammar schools. According to the Czech National Educational Program for Gymnasiums among other things the following topics are taught in gymnasiums in the Czech Republic within mathematics, (The Ministry of Education).

Solid geometry (stereometry).

This traditional geometric area covers the study of three-dimensional Euclidean space. It includes the measurements of volumes of various solid figures such as pyramids, prisms, polyhedrons, cylinders, cones, truncated cones, or spheres. Usually only regular solids are considered. When we solve some tasks of solid geometry a very important and crucial thing is the correct visualization of three-dimensional situation in two dimensions. According to this visualization we usually search for the solution. From these reasons the fundamental techniques of projections are taught. Here the *oblique projections*, (Carlbom & Paciorek, 1978), are used in most cases in gymnasiums in the Czech Republic. An oblique projection is a simple type of parallel projection which produces two-dimensional images with the specific properties – parallel lines are projected into parallel lines and if a polyhedron is projected, usually some its face or faces are parallel to the image plane (then these faces are projected in true shapes and sizes and are not distorted). Very typical task of solid geometry is to determine the size of an angle formed by two rays for example two adjacent edges or face and space diagonals of some polyhedron. A similar task is to find the size of a *dihedral angle*, i.e. the angle between two intersecting plane. We can consider two adjacent faces of some polyhedron but also some cutting planes of this polyhedron. Another typical task is to compute the size of an angle between a plane and intersecting straight line. Again the arbitrary combination of edges, faces or more general inputs can be considered. To understand these situations in three-dimensional space requires correct visualization. Students solve these tasks depicted in the plane by sketching. The main problem is the correct projection of three-dimensional situation and the depiction of known inputs. According to the correct visualization mathematical equations can be determined or we can solve the problem using synthetic geometry, i.e. using synthetic proofs of geometric theorems.

Example 1: Given a cube *ABCDEFGH*, find the size of a dihedral angle between the face *ACG* and the face *BCH*. An example is solved graphically in an oblique projection in Figure 1. It can be proved that the angle is 60°.

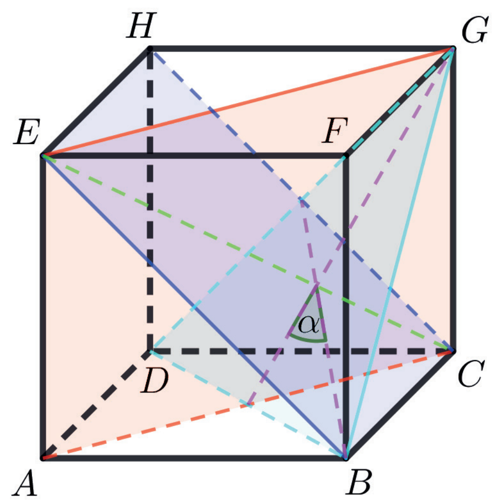

Figure 1: A dihedral angle between the face *ACG* and the face *BCH* depicted in an oblique projection, the situation is viewed from above from the right.

Cross section of solids.

Within solid geometry the specific topic we are dealing with is the *cross section of solids*. A cross section is the non-empty intersection of a solid figure with a plane. A plane containing a cross section of the solid is usually referred to as a *cutting plane*, (Pottman et al., 2007). The typical task for students is to determine the cross section of polyhedron if the cutting plane is given by three non-collinear points. The projection of three-dimensional situation is required again if we solve the task graphically. There are several basic steps in solving a cross section of solids based on the fundamental geometric statements regarding the intersection of three planes. Three non-coincident planes can intersect in one point; each plane cuts the other two in a line; two parallel planes and the other cuts each in a line; three planes intersect in a line; or three planes can be parallel.

Example 2: Given a cube *ABCDEFGH*, find the cross section of a cube by a plane *XYZ*, if points *X*, *Y*, *Z* are the centers of the edges *AB*, *CG*, *EH*; respectively. An example is solved graphically in an oblique projection in Figure 2. In this case the cross section is a regular hexagon.

Transformations of two- and three-dimensional shapes in three-dimensional space.

Finally, very important topics to study in three-dimensional space are transformations. The transformations are plane symmetries, point reflections, axial symmetries, rotations or translations. The compositions of these transformations can be studied too.

Example 3: Given a cube *ABCDEFGH*, find the reflection of a cube in a point reflection with a center *G* (a vertex of a cube). An example is solved graphically in an oblique projection in Figure 3.

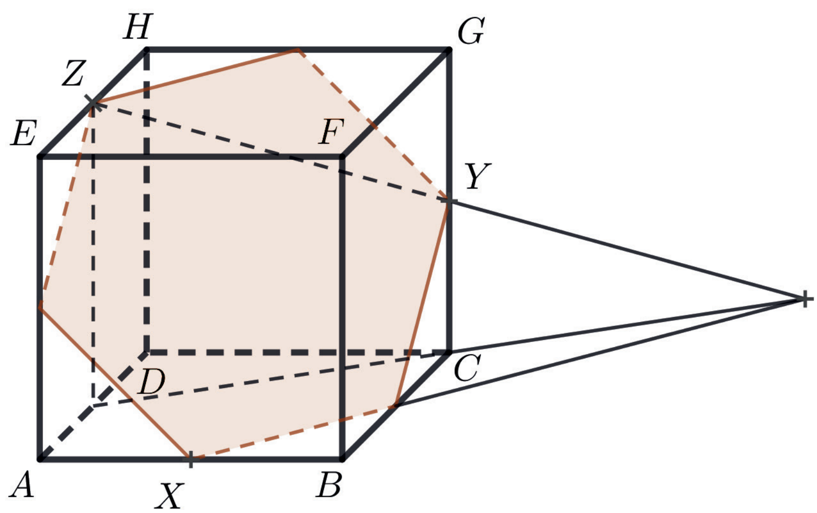

Figure 2: The cross section of a cube by a plane *XYZ* depicted in an oblique projection, the situation is viewed from above from the right.

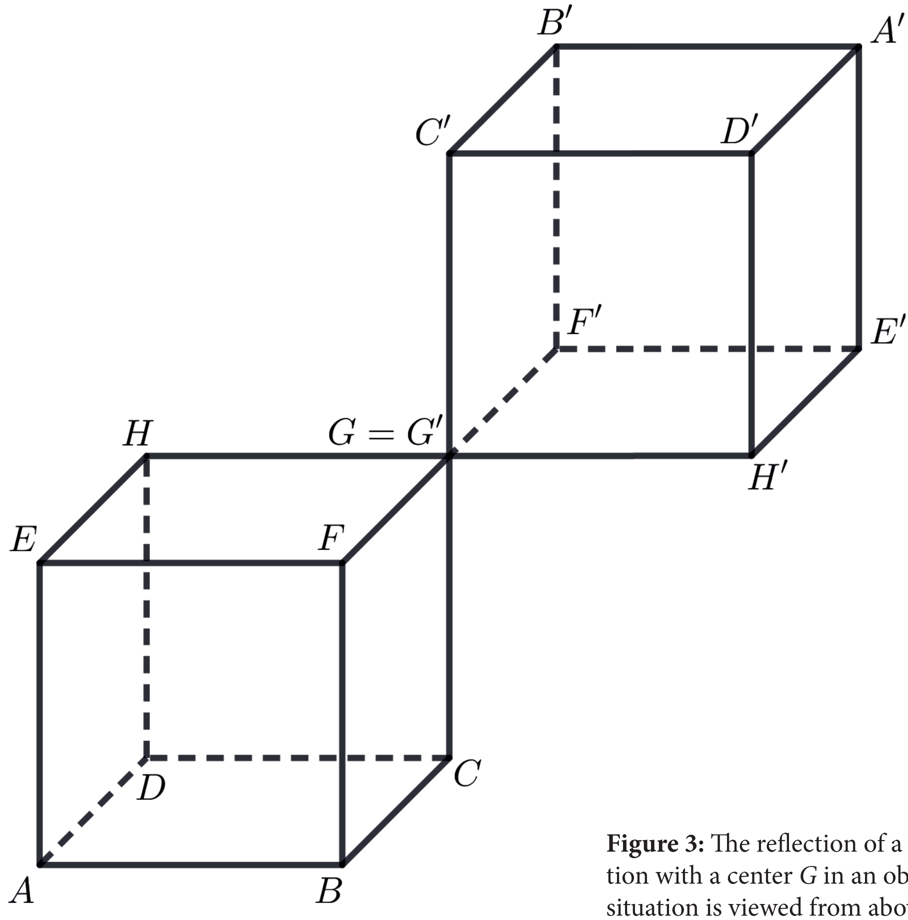

Figure 3: The reflection of a cube in a point reflection with a center *G* in an oblique projection, the situation is viewed from above from the right.

IDENTIFYING STUDENTS' PROBLEMS IN SPATIAL GEOMETRY

Let's discuss the main students' incomprehension these concrete topics. As has been already pointed out, we build on these topics in courses of Euclidean geometry, descriptive geometry and computational geometry at the faculty. This can be very difficult if students don't understand the fundamentals of spatial geometry. During my eight-year academic career at the Faculty of Mathematics and Physics I have been teaching more than 100 students within geometry lectures and seminars. Since I started teaching I have been collecting the data for the qualitative comparative analysis of the spatial skills of my students which is planned to be done in the future. I have identified main students' problems in geometry so far which are the following:

- the visualization of the three-dimensional situation - sketching of the three-dimensional situation in a parallel projection for instance in an oblique projection (for example an incorrect foreshortening of line segments which are not parallel to the image plane),
- understanding a two-dimensional image – the result of a parallel projection (for example an incorrect determination of true lengths, sizes and shapes which are distorted in the projection),

- the students are usually able to sketch the three-dimensional situation in the plane if it is viewed from above from the right (customarily used in the textbooks), Figure 4, the problem appears when the situation is viewed from above from the left or from below from the left and from the right, Figure 5.

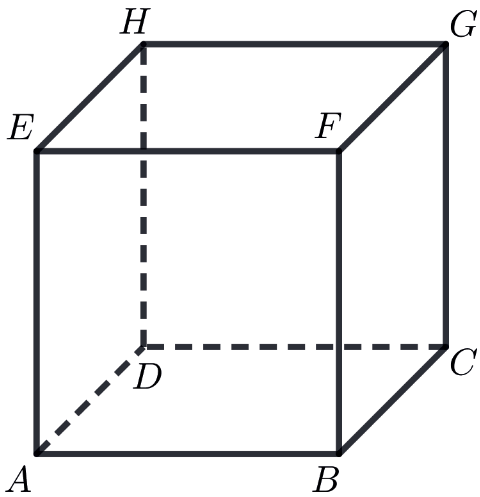

Figure 4: The cube viewed from above from the right. The image plane is considered to be vertical, it can be the screen.

The third mentioned problem appears very often. Students are able to solve the example 3 using the projection in Figure 3 but they have problems finding the solution if the same three-dimensional situation is viewed from below. The view from above is commonly used in the textbook but we shouldn't restrict only to it. We can also distinguish the position of the image plane. In this paper we can consider that the image plane is vertical (it can be the screen or the blackboard) in all figures.

SUGGESTED EXERCISES AND ACTIVITIES

On the ground of these findings mentioned above I am practicing specific topics in spatial geometry. To improve an understanding of spatial geometry I use 3D computer models together with physical models of solid figures and three-dimensional situations as a visual aid when students solve the geometric tasks depicted in the plane. The students are much more successful if the 3D computer model or physical model is provided. Also according to Katsioloudis, Jovanovic, & Jones (2014) using

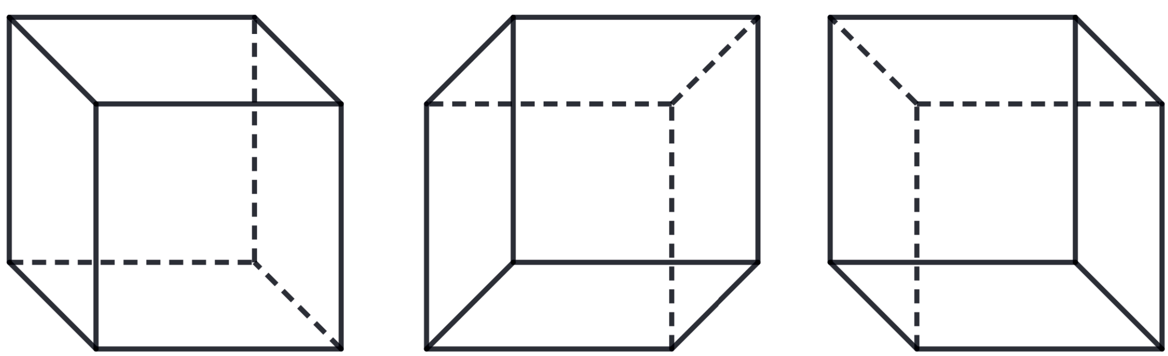

Figure 5: The cube viewed from above from the left, from below from the left, and from below from the right. The image plane is considered to be vertical in all cases, it can be the screen.

3D solid models as visualizations aids for Industrial Technology and Technology Education courses has great potential to improve spatial visualization skills. Besides the other software I am using mathematics and geometry dynamic software *GeoGebra*.

I found very important to practice hand sketching of solid figures and three-dimensional situations in various graphical projections using arbitrary views - from above and from below. Why should we restrict ourselves only to the view from above when in everyday life we are observing objects which surround us from the various viewpoints?

CONCLUSION AND FUTURE WORK

I presented main students' problems in geometry courses which have been identified during my eight-year teaching at the Faculty of Mathematics and Physics. I suggested several activities which are helping my students to achieve better results in understanding of spatial geometry.

According to my teaching experiences and identified students' problems I am modifying the most common tests for assessing spatial abilities such as the Mental Rotation Tests (Vanderberg & Kuse, 1978) or the Purdue Spatial Visualization Test-Visualization of Rotations (Guay, 1977). I plan to ask student newcomers to complete these tests in the winter semester 2018 as an entrance exam to prove or to disprove my hypotheses. According to the results I will focus on improvement of certain parts of spatial skills of my students through 3D computer modeling, physical models and practicing spatial skills on concrete examples in the winter semester. The similar testing is planned to be done at the end of the winter semester after geometry course.

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GEOMETRIA PRZESTRZENNA W SZKOLNICTWIE ŚREDNIM: KSZTAŁCENIE UMIEJĘTNOŚCI PRZESTRZENNYCH ORAZ ROZPOZNAWANIE PROBLEMÓW UCZNIÓW W DZIEDZINIE GEOMETRII PRZESTRZENNEJ

ABSTRAKT

Niniejszy artykuł dotyczy nauczania geometrii przestrzennej w szkołach średnich oraz związanych z nim sugestii odnoszących się do kursów geometrii prowadzonych w szkołach wyższych. Skupimy się przede wszystkim na zagadnieniach z dziedziny geometrii stanowiących przedmiot nauczania w Republice Czeskiej, tzn. zagadnieniach z zakresu geometrii przestrzennej, przekroju brył, przekształceń dwu- i trójwymiarowych figur geometrycznych w przestrzeni trójwymiarowej oraz wielu innych. Przedstawimy kilka przykładów typowych zadań geometrycznych na poziomie kształcenia średniego oraz zaawansowanych zagadnień realizowanych podczas kursów geometrii w szkołach wyższych. Doświadczenia zdobyte podczas prowadzenia kursów geometrii (na Uniwersytecie Karola w Republice Czeskiej) przeznaczonych dla studentów rozpoczynających naukę na uniwersytecie wskazują na poważne braki w zakresie ich umiejętności przestrzennych. Zidentyfikujemy niektóre z konkretnych problemów studentów w dziedzinie geometrii przestrzennej oraz przedstawimy kilka koncepcji nowo zaproponowanych ćwiczeń i zajęć przeznaczonych zarówno dla uczniów szkół średnich, jak i studentów, które mogłyby przyczynić się do zmniejszenia poziomu tych braków. Podczas zajęć korzystamy z pomocy dynamicznego systemu GeoGebra powszechnie używanego w procesie kształcenia na całym świecie.

Słowa kluczowe: geometria przestrzenna; umiejętności przestrzenne; szkolnictwo średnie; GeoGebra