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COMPARATIVE GEOMETRY: BREAK BARRIERS BETWEEN MATH AND MATH PHOBICS

Motto:
The trait that students value most in their teacher is her ability to convince them of their own abilities.

We discuss some features of school mathematics as important factors of math phobia among the public. We try to highlight the difference between the two images of math, as viewed by math phobics and math enthusiasts like us. We give proposals to deal with the problem in mathematics education, and a more detailed description of a method called Comparative geometry on the plane and on the sphere. We add some concrete experiences and experiments in and out of the classroom. The target audience of the present paper is not confined to the experts of mathematics teaching, but to the educational community as a whole.

Keywords: math education, math phobia, motivation, attitudes towards mathematics, comparative geometry.

INTRODUCTION

Recent historical events highlight again the responsibility of Education in a democratic society. The right to vote has little meaning without adequate voter competence in complex decision making situations.

It is impossible to provide, within the framework of general education, all knowledge in mathematics, history, finance, etc. that is needed to deal with a difficult political or economical problem. The task is to strike the optimal balance in the student between self-control and self-confidence; to help him recognize false information, misuse of scientific data, or lying in the disguise of mathematical arguments. To educate men and women of independent minds constituting a coherent, tolerant and environmentally conscious society – this is the essence of all educational tasks.

Most scholars and practitioners agree upon the main objectives of math teaching. '*Think creatively*', '*Construct knowledge on your own*', '*Math is fun*', and other popular slogans express good intentions of creators and teachers of the curriculum.

Why, then, grave lack of applicants at math faculties? Why the decline of math competence during the school years? Why the scarcity and low standards of math questions (and even lower standards of answers) at quiz programs in the media? Why the fear of math as the hardest of all hard subjects in the school? Why does math anxiety prevail over math love among the public?

Who or what to blame for this? Part of the answer connects to the public image of mathematics, but it is also necessary to analyze different aspects of the educational system as a whole.

IMAGE OF MATHEMATICS

For us authors, Mathematics is one of the greatest achievements of the human mind, a composition of common sense and pure poetry, a fairy-tale about non-existent terms like points, lines or numbers, yet directly linked to the miracles of science and technology. It is not at all infallible, but always ready to learn from its faults, and turn them into advantage. It is about the adventures of free and brave thought in the pursuit of harmony and truth.

This is diametrically opposed to the images of boring formulas, hopeless exercises, and ancient names to memorize (and forget immediately after the exam) - without any relevance to the interest of present-day learners.

'Hard and boring, but useful'?

This argument, apparently in favor of math as a school subject, is in fact one of the motives of math phobia in and out of the school.

Modern information technology makes mathematical formulas and theorems directly accessible for the student, from elementary to highly advanced level. No wonder that young people raise queries about the necessity of memorizing facts which are at hand on their ICT devices in any moment. We do not deny the importance of teaching about traditional concepts. Children should be acquainted with the multiplication table – but in a quite different sense as it had been taught for centuries before the era of computers.

Besides, we have deep doubts about the grounds for admitting '*boring but useful*' subjects in the school. As remarked above, no school system can provide full knowledge in any subject, including any branch of mathematics. Therefore, the task is just the opposite of teaching about '*boring but useful*' topics, but on the contrary, to evoke eagerness for moments of revelation, satisfaction, and self-confidence via mathematical activity. '*Useful topics*' can not compensate the learner for boredom, anxiety and inferiority complexes.

We add a further remark on a widespread misunderstanding. The often-cited claim on '*real-life math*' in school does not mean that each exercise directly refers to buying in a shop or calculating monthly payment for a loan. '*Real-life math*' is a way of dealing with our experiences, solving the problems that we encounter in our life. By an apparently far-fetched comparison, we do not learn about Hamlet because fratricide is so frequent in '*real life*', but because the way Hamlet faces the problem and tries to set it right, is very close to our own, his 21st-century readers. Likewise, a '*real-life*' situation in mathematics means much more than to invent everyday applications of a mathematical concept. It is not the outer garment, but the true spirit of mathematics that makes the subject '*real-life*'.

HARD ISSUES IN MATH EDUCATION

Aneta Czerska, author of the project "Wonderful Child", raises the following questions (Czerska, 2015): Does the school kill mathematical talents? Do schoolbooks kill mathematical talents? Does the teacher kill mathematical talents? Does the curriculum kill mathematical talents? Does the child deliberately kill his/her mathematical talent? Do parents lose mathematical talents in their children?

A. Czerska states that these questions concern all kids, because mathematical talent can be developed in any student, in one form or the other. Logically, she also raises the question of responsibility for the creation of math phobia in the school.

Sian Beilock from Chicago University says that the responsibility lies with first grade teachers, mostly women in primary schools, because children seize fear and stereotypes from their teachers. This might be a sign that female teachers are not adequately prepared for the subject, and their math anxiety has a negative effect on schoolgirls' achievement. (Beilock, 2010).

Even a superficial view of the curriculum of elementary teachers-to-be confirms the impression that math education is of secondary importance at pedagogical faculties of our universities. Many primary teachers-to-be have strong aversion to the subject of mathematics. Research conducted by the Institute of Educational Research in 2012-2014 showed that one among five undergraduates was strikingly ignorant about fundamental concepts in mathematics. Instructor's ignorance is very likely to kill student's interest, and add to the negative image of math among the public (IBE, 2015).

This is but one side of the picture, however. In our opinion, the main cause is to be found in teacher training. Bertrand Russell in "The Principles of Mathematics" wrote: '*Mathematics… possesses… a beauty cold and austere, like that of sculpture…*' *(Russell, 1903),* and some college and university lecturers are still cherishing the antiquated image of math as *a 'cold and austere*' subject whose core message cannot and should not be distributed among the '*masses*'. This conception is often associated with particular disdain for education as such: '*Math teachers are not scholars, just craftsmen*', '*Pedagogy is not a science, but a handful of everyday practices*', and the like. (References are not available, because these views are distributed via personal communication.)

In our opinion, this conception is one of the greatest obstacles in distributing and (even more importantly) popularizing mathematics among the laymen. In the present article, our aim is to provide suggestions that may be of some help in this regard.

MOTIVATION OR DEMOTIVATION FOR MATH LEARNING IN THE SCHOOL?

Good motivation for math learning is one of the best treatments against math phobia. As such, it is in the focus of the Report "*Mathematics Education in Europe: Common Challenges and National Policies*": "It should be noted, however, that motivation for doing mathematics is not a stable learner characteristic but a dynamic, changeable feature. For example, the thematic report of the Czech School Inspectorate 2008 and the 2008 Scottish Survey of Achievement compared the motivation of students in different school years. Both reports conclude that student motivation declines throughout secondary school – a finding which highlights the important role of teachers and the teaching process in using diverse teaching methods and supporting learner motivation. (Another sign that it is not only primary school teachers to blame!) TIMSS results also confirm that fourth grade students have much more positive attitudes towards mathematics than eighth graders. On the average, 67 % of fourth grade students and only 39 % of eighth grade students had very positive attitudes towards mathematics in EU countries participating in TIMMS report." (Report, 2011).

What is the reason of this decline? How to stop or even reverse the process?

Phyllis L. Newbill in her dissertation "Instructional Strategies to Improve Women's Attitudes toward Science" says that an important aspect related to motivation and achievement is the impact of students' attitudes towards mathematics. Attitudes are psychological states which are made up of three components: a cognitive component, an emotional component, and a behavioral component. In the context of education, they are seen as personal factors that affect learning (Newbill, 2005).

Akinsola M. K. and Olowojaiye F. B. in their article "*Teacher instructional methods and student attitudes towards mathematics*" say that "students' positive attitudes towards mathematics, which may be enhanced through effective teaching strategies, can promote learning achievement". Negative feelings or anxiety, on the other hand, can become a barrier to achieving good learning outcomes (Akinsola, Olowojaiye, 2008). By contrast, mathematics anxiety is thus an affective, or emotional, state, which has been shown to impair student performance – we can read in (Zientek, Yetkiner & Thompson, 2010).

Another factor with strong impact on the positive attitude towards mathematics and motivation is students' self-belief in their own abilities – as stressed in the motto of this paper, and proved by many researches, for example (Pajares, Kranzler, 1995), (Pajares, Miller, 1994), (Pajares, Graham, 1999), (Hackett, Betz, 1989).

Another report on the problems of motivation titled "*Mathematics Education in Europe: Common Challenges and National Policies*" gives the following advice:

"Mathematics teaching at school should encourage students' motivation to actively participate in the learning process. The nature of the tasks and exercises used for instruction have a great influence on whether students feel challenged and interested in mathematics and therefore motivated to engage with the learning process."

Research on key influences on students' positive attitudes towards mathematics suggests that "teaching methods and tasks must be engaging, diversified and connected to students' everyday life." (Report, 2011)

WHAT AND HOW TO TEACH?

FAQs (Frequently Asked Questions) by students in the math class are: Why do we need to deal with this or that topic? Why do we need to prove a theorem? Why do we need to learn a mathematical theory?

For many students, the use of mathematics is confined to paying at the desk in a plaza, or planning renovations at home which may include some elementary geometric considerations and arithmetical operations. They do not feel (or rather their math classes do not make them feel) that mathematics gives them the opportunity to learn thinking, reasoning, and making decisions - skills that may prove of vital importance in many domains of life. And probably very few students associate mathematics with artistic beauty and harmony, in contrast with the famous bon mot of English mathematician G. H. Hardy: '*Beauty is the first test: there is no permanent place in the world for ugly mathematics.*' (Hardy, 1940)

In their follow-up reports after annual national math exams in Poland, specialists relate problems of math education visible from exam results, and give recommendations for the schools on possible ways of solving the problems.

One of the reports at the end of junior secondary school (16 y/o students) in 2015 highlights the following problems:

- Particular attention be paid to tasks that involve reasoning and argumentation where the level of execution has remained low in consecutive years. For example, correct geometric constructions that require independent formation of the problem were given by less than 35% of the students. Apparently, it is big problem for many of them to prove a statement or justify a thesis.

- Pupils often use awkward mathematical language in their reasoning, without well-thought-out strategy and order. They forget to summarize their results or to draw conclusions.
- They successfully deal with problems in the practical context, or tasks that can be solved by well-known and practiced algorithms. However, their acquired knowledge often lets them down in the face of an unexpected situation that requires a relevant new algorithm to apply.
- Sometimes they have difficulties with reading and analyzing the task.

Recommendations for schools:

- In practice, the school should use a greater number of tasks on justification and reasoning, develop and educate the language of expression, sustain good mental habits, encourage activity in problem solving situations.
- It is good challenge to give them unexpected tasks presented in a peculiar style, because tackling an unusual situation is an important skill in and out of mathematics. (Report1, 2015)

The report from math exam at the end of secondary school (19 y/o students) highlights the following problems:

- Students encounter serious difficulties with tasks that require reasoning about justifying the truth of a statement or describing a property of mathematical objects.
- The most difficult tasks are those which require a complex strategy built on several consecutive stages.

Recommendations:

- It is extremely important to shape the students aware that they are expected to provide not only a string of equations or calculations, but also detailed explanation of dependencies between the steps, and description and verbal justification of their reasoning.
- Special regard be given to multi-step processes in mathematics, because the ability to develop and carry out a logic over successive activities is of very high value, and math is one of the most appropriate subjects to develop this skill. (Report2, 2015)

The above list displays some important factors of math phobia. Different ways have been developed by teachers and educators to overcome these problems. In what follows, we try to contribute to these efforts by a method of teaching geometry at various levels of education.

COMPARATIVE GEOMETRY BETWEEN THE PLANE AND THE SPHERE

To involve math students into active participation instead of passive acceptance (often resistance…), we must offer them interesting and challenging decision making situations at their own level of competence.

The problem is that each branch of school math (arithmetic, geometry, etc.) is based on one fixed way of thinking or (in the language of mathematics) on one system of axioms. It begins with a pack of ‘*obvious*’ assumptions without any dispute or digression on the student’s side. Only after accepting all these can the learner debate any statement, or create new findings on his own.

Math phobia often originates from the learner’s impatience to wait for a few interesting moments in return for many boring hours (or weeks…).

We propose changing methodology of math teaching by introducing two or more different approaches in the same branch of mathematics, teaching about different systems of axioms simultaneously. This idea can be applied in and out of mathematics, including subjects in human and social sciences, as history, linguistics, and so on.

In the present article we restrict to Comparative geometry between the plane and the sphere.

How can we teach it? The fundamental method on which the project is built is direct experimentation by the student, mainly on three-dimensional, palpable models, and also on virtual models and games. Apart from well-known devices in plane geometry, we apply special construction tools for spherical geometry, such as the spherical ruler, compass and protractor on the surface of a plastic sphere. Many introductory experiments can be carried out with everyday objects, as balls, globes, or round-shaped fruits.

Both of us have taught students of different age groups and very diverse prerequisite knowledge in mathematics, from schoolchildren in the elementary school to future kindergarten teachers. Our experience has been that direct experimentation in the math class has an amazingly strong impact on students' relation to mathematics. Profound indifference or even dislike can be transformed into positive emotions, even enthusiasm, towards the subject.

EXPERIENCE AND UNDERSTANDING

Another advantage of Comparative geometry is that '*the waiting period*' for interesting moments and decision situations becomes much shorter in many cases.

In our books on comparative geometry (Lénárt, 1996), (Rybak, Lénárt, 2013) we use the problem-based method to deal with a given topic. Our schema of students' investigation consists of the following stages:

- Set the problem.
- Make construction on the plane.
- Investigate on the plane,
- Make a guess: Predict the outcome of the same construction on the sphere.
- Perform the construction on the sphere.
- Investigate on the sphere.
- Compare the results of your investigations on the plane and on the sphere. Make conclusions.

Two decades of experience have shown that this structure is of good use both for students and teachers to discover a new world of geometry, as compared with traditional concepts of Euclidean geometry of the plane. Also it helps students to become more active and creative in math learning.

"OBJECTIONS"

There are many of them, but much more on the side of teachers than the kids. Most children are enthralled by the opportunity to draw on an orange or on a plastic sphere; to experiment with a spherical compass or ruler; to go into serious dispute about the nature of the point, or the number of centers of a circle, or a concave triangle on the sphere… Truth be told, it is much harder to make teachers catch fire than their students.

Some of the most frequent objections:

- 'Geometry is an application of the theory of transformations. As such, it does not deserve much attention.'
- 'I did not learn much about geometry myself, even less about non-Euclidean geometry – how could I teach it?'
- 'Spherical geometry cannot be taught to kids who hardly know Theorem of Pythagoras on the plane.'

- ‘Geometry requires a kind of creativity which an average student does not have. It is only for students of high ability.’
- (*The Big Objection*) ‘It might be a nice topic, but unfortunately, *I have no time for it!* Curriculum is so crowded; students must be prepared for entrance examinations!’

Two remarks to the ‘*Pythagoras*’ and ‘*No time!*’ paragraphs:

As for Pythagoras, the question is *what* we mean by ‘knowing the Theorem of Pythagoras’. Does correct memorizing of the text ‘*a* squared plus *b* squared …’ guarantee correct application of the theorem if one of the legs, not the hypotenuse, is labeled *c*? An even more important question: Do students *like* it? Do they appreciate the power and elegance of this theorem?

We ask our students: ‘*Is Pythagoras’ Theorem valid on the sphere? What is this theorem all about, anyway? Can this theorem, not proved, just be transferred from the plane to the sphere?*’ These questions turn the recitation of an ancient theorem into a thought-provoking game between two geometries.

As for time constraints, we have time for what we want to have time for. So the question is whether the whole enterprise takes more time than it really deserves. Again, the answer depends on the teacher’s goals. If we want to make our students recite a definition, then all this activity is superfluous or even harmful. However, if our aim is to make them really understand the meaning of ‘*straight*’, ‘*circle*’, ‘*polygon*’ or ‘*square*’, then, astonishingly, comparative geometry is not a loss, but a gain of time!

DESCRIPTION OF A TEACHING EXPERIMENT WITH SIXTH GRADERS

Adrienn Hege, former student of Eötvös Loránd University in Budapest, Hungary, Faculty of Early Education, now an elementary school teacher in Budapest, wrote master thesis titled “*The concept of angle in the elementary school introduced by the method of comparative geometry on the plane and on the sphere*”. In her thesis she described the experiment: a five-lesson research in comparative geometry with three groups of 12-year-old six-graders.

- Group A was specialized in math with six lessons per week – 10 girls, 12 boys.
- Group B with only three maths lessons per week represented ‘average’ knowledge in and affection towards the subject – 12 girls, 11 boys.
- Control group Z *(‘Zene’* is Hungarian for ‘music’) had special music curriculum but less demanding syllabus in math – 18 girls, 7 boys.

The average mark in mathematics from the previous school year was about the same in groups A and Z, but half a mark lower in group B.

Adrienn conducted five lessons about comparative geometry in groups A and B, another teacher conducted lessons only about angles on the plane in group Z.

The content of the five lessons was the following:

- Lesson 1: Simplest geometric element on the plane and on the sphere. Point and straight line. Into how many parts does one point on the straight line divide the line? And two points? Which part is finite/infinite? How many straight lines can be drawn through two points? Mutual position of two straight lines. Parallel and perpendicular lines. Point and opposite point. Construct a comparison table between the plane and the sphere.
- Lesson 2: Measuring distance on the two surfaces. What is measurement of distance good for? Tools for measuring distance. Exercises to solve on the plane and on the sphere. Extending the comparison table.
- Lesson 3: The concept of angle. Angle regions on the plane and on the sphere. Properties

of the spherical biangle. Methods and tools for measuring angles. Unigon and biangle (digon).

- Lesson 4: The concept of triangle. Sum of interior angles in a triangle on the plane and on the sphere.
- Lesson 5: Surprises on the sphere. Pentagramma mirificium (Napier).

During the first four lessons, group Z had their routine geometry program on the concepts of distance, angle, and polygons in Euclidean plane geometry. Groups A and B studied the same concepts via comparative geometry on the two surfaces.

The research hypothesis was: Comparing plane and sphere helps in understanding concepts of planar geometry and causes higher interests in mathematics by enabling individual and creative action. Improving the process of acquiring knowledge of elementary geometry (on the plane) by examining two different systems of geometry was especially stressed.

The last lesson was assigned to writing test papers in 40 minutes. Each group had the same 12 exercises connected exclusively to Euclidean plane geometry. Because of time constraints, they were not expected to work through all exercises, but to choose the ones they preferred to solve.

Examples of exercises:

- On arms of an angle measure out lengths 10 cm from the vertex and connect ends of segments that you received. Measure the length of the new segment. Is it true that after doubling the angle we will receive two times longer segment?
- Can three straight lines on the plane be constructed so that the first is perpendicular to the second, second to third, third to first? Can four straight lines on the plane arranged so that the first is perpendicular to the second, second to third, third to fourth, and fourth to first?
- Given a square and its diagonals, what is the measure of angles in the four small triangles in the inside of the square?
- Can angle measure be determined with a segment of a straight line?

On the diagram below we can see the percents of solutions of the exercise "Can three straight lines on the plane be constructed so that the first is perpendicular to the second, second to third, third to first?" in the three groups:

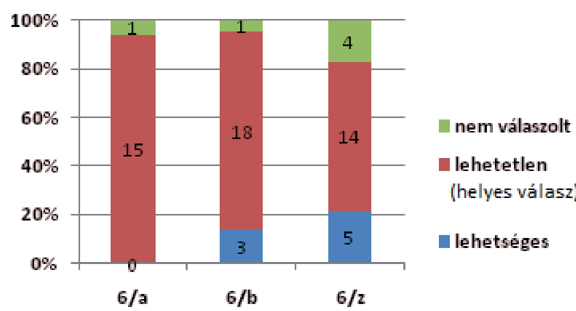

Fig. 1. Correct and incorrect students' solutions in percents [*Source:* Adrienn Hege *A szög fogalmának vizsgálata általános iskolában a sík- és gömbgeometria összehasonlító módszerével (The concept of angle in the elementary school introduced by the method of comparative geometry on the plane and on the sphere),* master thesis]

Green colour means no answer (nem válaszolt), red – good answer (lehetetlen, such construction is impossible), blue – bad answer (lehetséges). Explanations of pupils from Group B: „It is impossible on the plane, but possible in space", „It is impossible to draw the third straight line in this manner, because there will be acute angles with two straight lines."

The following table contains the results of solving all exercises in test (in percents):

Tab. 1. Comparing the solutions of students of three classes for assignments on the test

Ex. No		Group 6a	Group 6b	Group 6z
1.	No solution (in %)	0	0	0
	Good solution (in %)	25	50	0
	Bad solution (in %)	75	50	100
2.	No solution (in %)	0	0	9
	Good solution (in %)	23	32	17
	Bad solution (in %)	77	68	74
3.	No solution (in %)	19	4	9
	Good solution (in %)	62	48	21
	Bad solution (in %)	19	48	70
4.	No solution (in %)	Type of exercise required another kind of assessment than „good solution" or „bad solution".		
	Good solution (in %)			
	Bad solution (in %)			
5.	No solution (in %)	6	4	17
	Good solution (in %)	94	82	61
	Bad solution (in %)	0	14	22
6.	No solution (in %)	13	0	17
	Good solution (in %)	87	73	74
	Bad solution (in %)	0	27	9
7.	No solution (in %)	19	32	22
	Good solution (in %)	56	36	52
	Bad solution (in %)	25	32	26
8.	No solution (in %)	37	36	43
	Good solution (in %)	19	19	14
	Bad solution (in %)	44	45	43
9.	No solution (in %)	62	36	67
	Good solution (in %)	19	36	10
	Bad solution (in %)	19	28	23
10.	No solution (in %)	56	23	52
	Good solution (in %)	31	32	13
	Bad solution (in %)	13	45	35
11.	No solution (in %)	50	18	23
	Good solution (in %)	6	41	4
	Bad solution (in %)	44	41	73
12.	No solution (in %)	75	14	35
	Good solution (in %)	0	22	22
	Bad solution (in %)	25	64	43

Source: Anna Rybak based on Adrienn Hege *A szög fogalmának vizsgálata általános iskolában a sík- és gömbgeometria összehasonlító módszerével (The concept of angle in the elementary school introduced by the method of comparative geometry on the plane and on the sphere) master thesis*

SUMMARY OF EVALUATION

Most of "no answers" come from Class 6/z. Among other factors, this can also be explained by the fact that Class/a and Class/b participated in the sphere lessons, and they were more motivated than Class/z.

An interesting and thought-provoking finding is that the greatest difference between the half-year average and the test-paper average belongs to Class/b (1.2), in contrast with Class/a (0.86) or Class/z (0.87). It is shown in the following diagram:

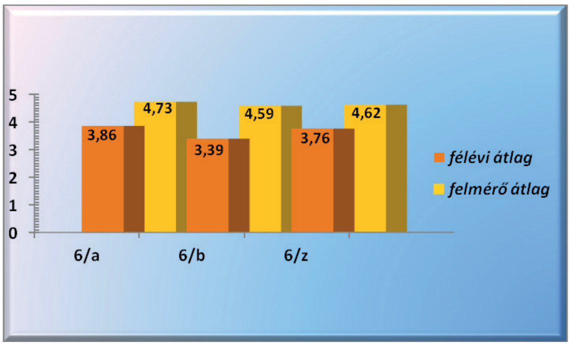

Fig. 2. Comparing the average of semester evaluations and the average of evaluations from the test [*Source:* Adrienn Hege *A szög fogalmának vizsgálata általános iskolában a sík- és gömbgeometria összehasonlító módszerével (The concept of angle in the elementary school introduced by the method of comparative geometry on the plane and on the sphere)* master thesis]

Average semester evaluations (félévi átlag) are marked with orange colour, average results of tests (felmérő átlag) with yellow colour.

According to our expectations, group A with its special math syllabus outperformed group Z; but, astonishingly, group B surpassed both A and Z in most cases. They provided better results and more detailed explanations, even if with more spelling mistakes. Apparently, they developed closer contact to reality, more confidence in their senses and skills than their peers. The usual disadvantages of meager socioeconomic background were outweighed by more flexible thinking and mature judgment. They were happy with their success, an unusual experience for them in math lessons. It is worth remarking that the difference between the average mark at the test paper and the average mark in math at the end of the term was +1.2 in group B, but only +0.86 in group A and +0.87 in group Z. This was another sign of greater inspiration for the '*weaker*' group B as compared with the other two groups. (Hege, 2011)

One possible explanation may be that this is a relatively new material, independently from the class (a, b, or z) that students attend. The knowledge that students have brought from home or from former school studies are less important in this case. All kids start from nearly the same line, so to speak. It may become clear that indifference in or dislike towards mathematics is rooted, not in lack of abilities, but lack of favourable circumstances. Kids who very rarely enjoy feeling of success in maths are positively influenced and encouraged by geometric experimentation.

DESCRIPTION OF EXPERIENCES IN NON-EXPERIMENTAL SCHOOL SITUATIONS

Anna Rybak conducted a lot of lessons and workshops with students in different age and teachers who wanted to learn something new in geometry, to use different didactic tools and experience new style of learning. After some such lessons students gave a feedback – they answered questions in the survey. The following table contains these questions and the most typical responses:

Table 2. Feedback from students after the lesson about spherical geometry

Question	Answers of students
What was the lesson about? What was the purpose of the lesson?	The lesson was about the sphere; The purpose was to make knowledge about sphere more familiar; About the geometric figures on the sphere; Drawing the figures on the sphere; Repeating the knowledge from the last lessons; I do not know.
What did you do during the lesson? Describe your actions.	Students indicated drawing on the sphere, constructing different figures, measuring the angles and calculating.
What did you learn at the lesson?	That it is impossible to draw square on the sphere; How to draw on the sphere; Triangle on the sphere has different properties; How to use spherical ruler; Something about figures on the sphere; To work with tools for measuring on the sphere; About the angles of drawn triangle: 90^o, 90^o, 90^o; I learnt that straight line on the sphere is a circle; Sum of angles in quadrilateral on the plane is 360^o , on the sphere more than 400^o. In the triangle sum of angles is more than 180^o; A lot of interesting things concerning sphere; That even on the sphere we can draw figures. On the sphere there are more degrees than on the plane (*original style of sentence*); Nothing.
In your opinion, what was at the lesson good?	Drawing on the sphere; Drawing and amusement.; Acquiring knowledge; Free lesson. Interesting lesson; Amusement. We did not need to calculate anything; Three angles of triangle equal 90^o. Ruler. Protractor. Compass; Work with the sphere means amusement; Interesting. New information. New tools; Nice atmosphere; I learned interesting things. I worked first time with such tools. Lesson was interesting and instructive; Education together with playing; Drawing figures and their calculating; Everything; Nothing.
In your opinion, what was at the lesson bad?	Too short lesson; Dirty hands; Difficulties with joining two hemispheres together.

Table 2. cont.

What would you change at the lesson if the lesson was to be repeated?	Students had no proposals of changes in the lesson.

Source: composed by Anna Rybak

Remarks of teacher conducting lesson (Anna Rybak):

• Functional strategy of teaching was used at the lesson. Students performed concrete activities (they worked with real objects), notional activities (they used their drawings that were representations of geometric figures) and abstract activities (they made calculations, conclusions, described properties of figures). Students used different kinds of representations: enactive (sphere), iconic (drawings, images) and symbolic.

• Constructivistic approach was applied: students constructed knowledge about properties of figures on the sphere by themselves. Of course during such lessons students must be inspired by a teacher, who formulates problems and asked proper questions.

• Comparative method was used at the lesson.

• Three lessons on the same topic were conducted the same day with three groups of students in age 16 (third grade of junior secondary school with integration branches). All students (independently from their physical or mental disabilities) were doing manual activities, drawings and measurements. Discussion was conducted with the whole group on each lesson.

Conclusions from observation of students during the lessons and analysis of their feedback:

• It is possible to introduce comparative geometry into everyday math education without making changes in curriculum and without disturbance everyday work with students.

• Observation of students' work during the lesson and analysis of their responses show that the possibility of independent examining figures is attractive for them and using functional learning method and problem-solving method made the lesson different, more enjoyable. Students found such lesson as play joined with learning. Students appreciate learning through play.

• Students want to work with didactic tools. They want to perform concrete activities!

• The atmosphere of the discussion is well received by the students.

• The main knowledge about spherical geometry (which in many environments is found as difficult, even academic) may be constructed by students from average group in normal circumstances within not long time.

DESCRIPTION OF OUR EXPERIENCES AT FESTIVALS, PICNICS ETC. WORKING WITH GENERAL AUDIENCE OF DIFFERENT AGE GROUPS AND DIFFERENT PROFESSIONS.

Our experiences at public events of popularizing science (Festivals of Science, Scientific Picnics, Science Museums, etc.) show that this domain of knowledge is interesting and challenging for everybody. Children like very much to make free drawings or constructions on the sphere. They often exclaim: '*I am creating!*' What is more, even grown-up members of the family get involved into the game. Parents and grandparents reluctantly stop behind their enthusiastic kids: '*Why stop here? It is math, it is dead boring!*' they say. It is really uplifting an experience for us presenters to see how they gradually catch fire, how they turn from sufferers into listeners, then finally enjoyers of the scene: '*I never thought that geometry could be fun for me!*' '*If only I had been taught math in my school this way!*'

CONCLUSIONS

Our conviction is that beyond teaching about useful and necessary formulas and procedures in school mathematics, the central task is to break the barriers of indifference or hostility between the general public and the subject of mathematics; to convey the image of a warm, friendly, enjoyable activity, filled with the highest kind of artistic and poetic beauty that the human mind is capable of. '*Students are not vessels to be filled, but rather lamps to be lit.*' - Plutarch said (Plutarch, 110). To light a spark of interest in our students towards a beautiful subject, to give self-confidence, independent thinking, trust in their own creative power, and boost for encounters with mathematics in their life – this is the challenge for us teachers when we are honored by the attention of young people in our classrooms.

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GEOMETRIA PORÓWNAWCZA: PRZEŁAMYWANIE BARIER POMIĘDZY MATEMATYKĄ I FOBIĄ MATEMATYCZNĄ

STRESZCZENIE:

W artykule omawiamy niektóre cechy szkolnej matematyki jako ważne czynniki powstawania fobii matematycznej w społeczeństwie. Staramy się podkreślić różnicę między dwoma obrazami matematyki: tym oglądanym przez ludzi owładniętymi matematyczną fobią i tym oglądanym przez miłośników matematyki jak my. Przedstawiamy pewne propozycje przezwyciężenia problemów w edukacji matematycznej, i bardziej szczegółowy opis metody o nazwie Geometria porównawcza na płaszczyźnie i na sferze. Opisujemy też konkretne doświadczenia i eksperymenty prowadzone w klasie i poza klasą. Niniejsza praca nie jest kierowana tylko do ekspertów z zakresu nauczania matematyki, ale także do społeczności szkolnej jako całości.

Słowa kluczowe: edukacja matematyczna, fobia matematyczna, motywacja, postawy wobec matematyki, geometria porównawcza.